

ϕ -CLASSICAL PRIME SUBMODULESHOJJAT MOSTAFANASAB*, ESRA SENGELEN SEVIM,
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ABSTRACT. In this paper, all rings are commutative with nonzero identity. Let M be an R -module. A proper submodule N of M is called a *classical prime submodule*, if for each $m \in M$ and elements $a, b \in R$, $abm \in N$ implies that $am \in N$ or $bm \in N$. Let $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function where $S(M)$ is the set of all submodules of M . We introduce the concept of “ ϕ -classical prime submodules”. A proper submodule N of M is a ϕ -classical prime submodule if whenever $a, b \in R$ and $m \in M$ with $abm \in N \setminus \phi(N)$, then $am \in N$ or $bm \in N$.

1. INTRODUCTION

Throughout this paper all rings are commutative with nonzero identity and all modules are considered to be unitary. Anderson and Smith [3] said that a proper ideal I of a ring R is *weakly prime* if whenever $a, b \in R$ with $0 \neq ab \in I$, then $a \in I$ or $b \in I$. In [10], Bhatwadekar and Sharma defined a proper ideal I of an integral domain R to be *almost prime* (resp. *n-almost prime*) if for $a, b \in R$ with $ab \in I \setminus I^2$, (resp. $ab \in I \setminus I^n, n \geq 3$) either $a \in I$ or $b \in I$. This definition can obviously be made for any commutative ring R . Later, Anderson and Batanieh [2] gave a generalization of prime ideals which covers all the above mentioned definitions. Let $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. A proper ideal I of R is said to be ϕ -prime if for $a, b \in R$ with $ab \in I \setminus \phi(I)$, $a \in I$ or $b \in I$. Several authors have extended the notion of prime ideal to modules, see, for example [11, 15, 16]. Let M be a module over a commutative ring R . A proper submodule N of M is called *prime* if for $a \in R$ and $m \in M$, $am \in N$ implies that $m \in N$ or $a \in (N :_R M) = \{r \in R \mid rM \subseteq N\}$. Weakly prime submodules were introduced by Ebrahimi Atani and Farzalipour in [12]. A proper submodule N of M is called *weakly prime* if for $a \in R$ and $m \in M$ with $0 \neq am \in N$, either $m \in N$ or $a \in (N :_R M)$. Zamani [22] introduced the concept of ϕ -prime submodules. Let $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function where $S(M)$ is the set of all submodules of M . A proper submodule N of an R -module M is called ϕ -prime if $a \in R$ and $m \in M$ with $am \in N \setminus \phi(N)$, then $m \in N$ or $a \in (N :_R M)$. He defined the map $\phi_\alpha : S(M) \rightarrow S(M) \cup \{\emptyset\}$ as follows:

- (1) $\phi_\emptyset : \phi(N) = \emptyset$ defines prime submodules.
- (2) $\phi_0 : \phi(N) = \{0\}$ defines weakly prime submodules.
- (3) $\phi_2 : \phi(N) = (N :_R M)N$ defines almost prime submodules.
- (4) $\phi_n (n \geq 2) : \phi(N) = (N :_R M)^{n-1}N$ defines n -almost prime submodules.
- (5) $\phi_\omega : \phi(N) = \bigcap_{n=1}^{\infty} (N :_R M)^n N$ defines ω -prime submodules.

2010 *Mathematics Subject Classification*. Primary: 13A15; secondary: 13C99; 13F05.

Key words and phrases. Classical prime submodule, Weakly classical prime submodule, ϕ -Classical prime submodule.

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(6) $\phi_1 : \phi(N) = N$ defines any submodules.

Also, Moradi and Azizi [17] investigated the notion of n -almost prime submodules. A proper submodule N of M is called a *classical prime submodule*, if for each $m \in M$ and $a, b \in R$, $abm \in N$ implies that $am \in N$ or $bm \in N$. This notion of classical prime submodules has been extensively studied by Behboodi in [6, 7] (see also, [8], in which, the notion of classical prime submodules is named “weakly prime submodules”). For more information on classical prime submodules, the reader is referred to [4, 5, 9]. In [18], Mostafanasab et. al. said that a proper submodule N of an R -module M is called a *weakly classical prime submodule* if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $am \in N$ or $bm \in N$.

Let $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function where $S(M)$ is the set of all submodules of M . Let N be a proper submodule of M . Then we say that N is a ϕ -classical prime submodule of M if whenever $a, b \in R$ and $m \in M$ with $abm \in N \setminus \phi(N)$, then $am \in N$ or $bm \in N$. Clearly, every classical prime submodule is a ϕ -classical prime submodule. We defined the map $\phi_\alpha : S(M) \rightarrow S(M) \cup \{\emptyset\}$ as follows:

- (1) $\phi_\emptyset : \phi(N) = \emptyset$ defines classical prime submodules.
- (2) $\phi_0 : \phi(N) = \{0\}$ defines weakly classical prime submodules.
- (3) $\phi_2 : \phi(N) = (N :_R M)N$ defines almost classical prime submodules.
- (4) $\phi_n (n \geq 2) : \phi(N) = (N :_R M)^{n-1}N$ defines n -almost classical prime submodules.
- (5) $\phi_\omega : \phi(N) = \bigcap_{n=1}^\infty (N :_R M)^n N$ defines ω -classical prime submodules.
- (6) $\phi_1 : \phi(N) = N$ defines any submodules.

Throughout this paper $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ denotes a function. Since $N \setminus \phi(N) = N \setminus (N \cap \phi(N))$, for any submodule N of M , without loss of generality we may assume that $\phi(N) \subseteq N$. For any two functions $\psi_1, \psi_2 : S(M) \rightarrow S(M) \cup \{\emptyset\}$, we say $\psi_1 \leq \psi_2$ if $\psi_1(N) \subseteq \psi_2(N)$ for each $N \in S(M)$. Thus clearly we have the following order: $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \dots \leq \phi_{n+1} \leq \phi_n \leq \dots \leq \phi_2 \leq \phi_1$. Whenever $\psi_1 \leq \psi_2$, any ψ_1 -classical prime submodule is ψ_2 -classical prime.

An R -module M is called a *multiplication module* if every submodule N of M has the form IM for some ideal I of R , see [14]. Note that, since $I \subseteq (N :_R M)$ then $N = IM \subseteq (N :_R M)M \subseteq N$. So that $N = (N :_R M)M$. Let N and K be submodules of a multiplication R -module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R . The product of N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [1, Theorem 3.4], the product of N and K is independent of presentations of N and K . Moreover, for $m, m' \in M$, by mm' , we mean the product of Rm and Rm' . Clearly, NK is a submodule of M and $NK \subseteq N \cap K$ (see [1]). Let N be a proper submodule of a nonzero R -module M . Then the M -radical of N , denoted by $M\text{-rad}(N)$, is defined to be the intersection of all prime submodules of M containing N . If M has no prime submodule containing N , then we say $M\text{-rad}(N) = M$. It is shown in [14, Theorem 2.12] that if N is a proper submodule of a multiplication R -module M , then $M\text{-rad}(N) = \sqrt{(N :_R M)M}$.

In [19], Quatararo et. al. said that a commutative ring R is a u -ring provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals; and a um -ring is a ring R with the property that an R -module which is equal to a finite union of submodules must be equal to one of them. They show that every Bézout ring is a u -ring. Moreover, they proved that every Prüfer domain is a u -domain. Also, any ring which contains an infinite field as a subring is a u -ring, [20, Exercise 3.63].

Let M be an R -module and $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. It is shown (Theorem 2.11) that N is a ϕ -classical prime submodule of M if and only if for every ideals I, J of R and $m \in M$ with $IJm \subseteq N$ and $IJm \not\subseteq \phi(N)$, either $Im \subseteq N$ or $Jm \subseteq N$. It is shown (Theorem 2.14) that over a um -ring R , N is a ϕ -classical prime submodule of M if and only if for every ideals I, J of R and submodule L of M with $IJL \subseteq N$ and $IJL \not\subseteq \phi(N)$, either $IL \subseteq N$ or $JL \subseteq N$. It is proved (Theorem 2.30) that if N is a ϕ -classical prime submodule of M that is not classical prime, then $(N :_R M)^2 N \subseteq \phi(N)$. Let M_1, M_2 be R -modules and N_1 be a proper submodule of M_1 . Suppose that $\psi_i : S(M_i) \rightarrow S(M_i) \cup \{\emptyset\}$ be functions (for $i = 1, 2$) and let $\phi = \psi_1 \times \psi_2$. In Theorem 2.36 we prove that the following conditions are equivalent:

- (1) $N_1 \times M_2$ is a ϕ -classical prime submodule of $M_1 \times M_2$;
- (2) N_1 is a ϕ -classical prime submodule of M_1 and for each $r, s \in R$ and $m_1 \in M_1$ we have $rs m_1 \in \psi_1(N_1)$, $rm_1 \notin N_1$, $sm_1 \notin N_1 \Rightarrow rs \in (\psi_2(M_2) :_R M_2)$.

Let $R = R_1 \times R_2 \times R_3$ be a decomposable ring and $M = M_1 \times M_2 \times M_3$ be an R -module where M_i is an R_i -module, for $i = 1, 2, 3$. Suppose that $\psi_i : S(M_i) \rightarrow S(M_i) \cup \{\emptyset\}$ be functions such that $\psi(M_i) \neq M_i$ for $i = 1, 2, 3$, and let $\phi = \psi_1 \times \psi_2 \times \psi_3$. In Theorem 2.42 it is proved that if N is a ϕ -classical prime submodule of M , then either $N = \phi(N)$ or N is a classical prime submodule of M .

2. PROPERTIES OF ϕ -CLASSICAL PRIME SUBMODULES

Let M be an R -module, K be a submodule of M and $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. Define $\phi_K : S(M/K) \rightarrow S(M/K) \cup \{\emptyset\}$ by $\phi_K(N/K) = (\phi(N) + K)/K$ for every $N \in S(M)$ with $N \supseteq K$ (and $\phi_K(N/K) = \emptyset$ if $\phi(N) = \emptyset$).

Theorem 2.1. *Let M be an R -module and $K \subseteq N$ be proper submodules of M . Suppose that $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function.*

- (1) *If N is a ϕ -classical prime submodule of M , then N/K is a ϕ_K -classical prime submodule of M/K .*
- (2) *If $K \subseteq \phi(N)$ and N/K is a ϕ_K -classical prime submodule of M/K , then N is a ϕ -classical prime submodule of M .*
- (3) *If $\phi(N) \subseteq K$ and N is a ϕ -classical prime submodule of M , then N/K is a weakly classical prime submodule of M/K .*
- (4) *If $\phi(K) \subseteq \phi(N)$, K is a ϕ -classical prime submodule of M and N/K is a weakly classical prime submodule of M/K , then N is a ϕ -classical prime submodule of M .*

Proof. (1) Let $a, b \in R$ and $m \in M$ be such that $ab(m+K) \in (N/K) \setminus \phi_K(N/K)$. It follows that $abm \in N \setminus \phi(N)$, that gives $am \in N$ or $bm \in N$. Therefore $a(m+K) \in N/K$ or $b(m+K) \in N/K$.

(2) Let $a, b \in R$ and $m \in M$ be such that $abm \in N \setminus \phi(N)$. Then $ab(m+K) \in (N/K) \setminus \phi_K(N/K) = (N/K) \setminus (\phi(N)/K)$. Hence $a(m+K) \in N/K$ or $b(m+K) \in N/K$, and so $am \in N$ or $bm \in N$.

(3) Let $a, b \in R$ and $m \in M$ be such that $0 \neq ab(m+K) \in (N/K)$. Hence $abm \in N \setminus \phi(N)$, because $\phi(N) \subseteq K$. Thus $am \in N$ or $bm \in N$. Therefore $a(m+K) \in N/K$ or $b(m+K) \in N/K$.

(4) Let $a, b \in R$ and $m \in M$ be such that $abm \in N \setminus \phi(N)$. Note that $\phi(K) \subseteq \phi(N)$ implies that $abm \notin \phi(K)$. If $abm \in K$, then $am \in K \subseteq N$ or $bm \in K \subseteq N$,

since K is ϕ -classical prime. Now, assume that $abm \notin K$. So $0 \neq ab(m+K) \in N/K$. Therefore, since N/K is a weakly classical prime submodule of M/K , either $a(m+K) \in N/K$ or $b(m+K) \in N/K$. Thus $am \in N$ or $bm \in N$. \square

Corollary 2.2. *Let N be a proper submodule of M and $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. Then N is a ϕ -classical prime submodule of M if and only if $N/\phi(N)$ is a weakly classical prime submodule of $M/\phi(N)$.*

Theorem 2.3. *Let M be an R -module and N be a proper submodule of M . Suppose that $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ and $\psi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be two functions.*

- (1) *If N is a ϕ -classical prime submodule of M , then $(N :_R m)$ is a ψ -prime ideal of R for every $m \in M \setminus N$ with $(\phi(N) :_R m) \subseteq \psi((N :_R m))$.*
- (2) *If $(N :_R m)$ is a ψ -prime ideal of R for every $m \in M \setminus N$ with $\psi((N :_R m)) \subseteq (\phi(N) :_R m)$, then N is a ϕ -classical prime submodule of M .*

Proof. (1) Suppose that N is a ϕ -classical prime submodule. Let $m \in M \setminus N$ with $(\phi(N) :_R m) \subseteq \psi((N :_R m))$, and $ab \in (N :_R m) \setminus \psi((N :_R m))$ for some $a, b \in R$. Then $abm \in N \setminus \phi(N)$. So $am \in N$ or $bm \in N$. Hence $a \in (N :_R m)$ or $b \in (N :_R m)$. Consequently $(N :_R m)$ is a ψ -prime ideal of R .

(2) Assume that $(N :_R m)$ is a ψ -prime ideal of R for every $m \in M \setminus N$ with $\psi((N :_R m)) \subseteq (\phi(N) :_R m)$. Let $abm \in N \setminus \phi(N)$ for some $m \in M$ and $a, b \in R$. If $m \in N$, then we are done. So we assume that $m \notin N$. Hence $ab \in (N :_R m) \setminus \psi((N :_R m))$ implies that either $a \in (N :_R m)$ or $b \in (N :_R m)$. Therefore either $am \in N$ or $bm \in N$, and so N is ϕ -classical prime. \square

Corollary 2.4. *Let R be a ring and $\psi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Then I is a ψ -prime ideal of R if and only if ${}_R I$ is a ψ -classical prime submodule of ${}_R R$.*

Proof. First, note that ${}_R I$ is a ψ -prime submodule of ${}_R R$ if and only if I is a ψ -prime ideal of R . Now, apply part (1) of Proposition 2.5. Conversely, let ${}_R I$ be a ψ -classical prime submodule of ${}_R R$. Notice that $(\psi(I) :_R 1) = \psi((I :_R 1)) = \psi(I)$. Then by Theorem 2.3(1), $(I :_R 1) = I$ is a ψ -prime ideal of R . \square

Darani and Soheilnia [21] generalized the concept of prime submodules of a module over a commutative ring as follows: Let N be a proper submodule of an R -module M . Then N is said to be a *2-absorbing submodule of M* if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$. Let N be a proper submodule of an R -module M . Then N is said to be a *ϕ -2-absorbing submodule of M* if whenever $a, b \in R$ and $m \in M$ with $abm \in N \setminus \phi(N)$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$, see [13].

Proposition 2.5. *Let N be a proper submodule of an R -module M . Suppose that $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ and $\psi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be two functions.*

- (1) *If N is a ϕ -prime submodule of M , then N is a ϕ -classical prime submodule of M .*
- (2) *If N is a ϕ -classical prime submodule of M , then N is a ϕ -2-absorbing submodule of M . The converse holds if in addition $(N :_R M)$ is a ψ -prime ideal of R and $\psi((N :_R M)) \subseteq (\phi(N) :_R m)$.*

Proof. (1) Assume that N is a ϕ -prime submodule of M . Let $a, b \in R$ and $m \in M$ such that $abm \in N \setminus \phi(N)$. Therefore either $bm \in N$ or $a \in (N :_R M)$. The first case leads us to the claim. In the second case we have that $am \in N$. Consequently N is a ϕ -classical prime submodule.

(2) It is evident that if N is ϕ -classical prime, then it is ϕ -2-absorbing. Assume that N is a ϕ -2-absorbing submodule of M and $(N :_R M)$ is a ψ -prime ideal of R . Let $abm \in N \setminus \phi(N)$ for some $a, b \in R$ and $m \in M$ such that neither $am \in N$ nor $bm \in N$. Then $ab \in (N :_R M)$. Since $abm \notin \phi(N)$, then $\psi((N :_R M)) \subseteq (\phi(N) :_R m)$ implies that $ab \notin \psi((N :_R M))$. Therefore, either $a \in (N :_R M)$ or $b \in (N :_R M)$. This contradiction shows that N is ϕ -classical prime. \square

Definition 2.6. Let N be a proper submodule of a multiplication R -module M and $n \geq 2$. Then N is said to be *n -potent classical prime* if whenever $a, b \in R$ and $m \in M$ with $abm \in N^n$, then $am \in N$ or $bm \in N$.

Proposition 2.7. Let M be a multiplication R -module. If N is an n -almost classical prime submodule of M for some $n \geq 2$ and N is a k -potent classical prime for some $k \leq n$, then N is a classical prime submodule of M .

Proof. Assume that N is an n -almost classical prime submodule of M . Let $abm \in N$ for some $a, b \in R$ and $m \in M$. If $abm \notin N^k$, then $abm \notin N^n$. In this case, we are done since N is an n -almost classical prime submodule. So assume that $abm \in N^k$. Hence we get $am \in N$ or $bm \in N$, since N is a k -potent classical prime submodule of M . \square

Proposition 2.8. Let M be a cyclic R -module and $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. Then a proper submodule N of M is a ϕ -prime submodule if and only if it is a ϕ -classical prime submodule.

Proof. By Proposition 2.5(1), the “only if” part holds. Let $M = Rm$ for some $m \in M$ and N be a ϕ -classical prime submodule of M . Suppose that $rx \in N \setminus \phi(N)$ for some $r \in R$ and $x \in M$. Then there exists an element $s \in R$ such that $x = sm$. Therefore $rx = rsm \in N \setminus \phi(N)$ and since N is a ϕ -classical prime submodule, $rm \in N$ or $sm \in N$. Hence $r \in (N :_R M)$ or $x \in N$. Consequently N is a ϕ -prime submodule. \square

Theorem 2.9. Let $f : M \rightarrow M'$ be an epimorphism of R -modules and let $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ and $\phi' : S(M') \rightarrow S(M') \cup \{\emptyset\}$ be functions. Then the following conditions hold:

- (1) If N' is a ϕ' -classical prime submodule of M' and $\phi(f^{-1}(N')) = f^{-1}(\phi'(N'))$, then $f^{-1}(N')$ is a ϕ -classical prime submodule of M .
- (2) If N is a ϕ -classical prime submodule of M containing $\text{Ker}(f)$ and $\phi'(f(N)) = f(\phi(N))$, then $f(N)$ is a ϕ' -classical prime submodule of M' .

Proof. (1) Since f is epimorphism, $f^{-1}(N')$ is a proper submodule of M . Let $a, b \in R$ and $m \in M$ such that $abm \in f^{-1}(N') \setminus \phi(f^{-1}(N'))$. Since $abm \in f^{-1}(N')$, $abf(m) \in N'$. Also, $\phi(f^{-1}(N')) = f^{-1}(\phi'(N'))$ implies that $abf(m) \notin \phi'(N')$. Thus $abf(m) \in N' \setminus \phi'(N')$. Hence $af(m) \in N'$ or $bf(m) \in N'$ and thus $am \in f^{-1}(N')$ or $bm \in f^{-1}(N')$. So, we conclude that $f^{-1}(N')$ is a ϕ -classical prime submodule of M .

(2) Let $a, b \in R$ and $m' \in M'$ such that $abm' \in f(N) \setminus \phi'(f(N))$. Since f is epimorphism, there exists $m \in M$ such that $m' = f(m)$. Therefore $f(abm) \in f(N)$ and so $abm \in N$, because $\text{Ker}(f) \subseteq N$. Since $\phi'(f(N)) = f(\phi(N))$, then $abm \notin \phi(N)$. Hence $abm \in N \setminus \phi(N)$. It implies that $am \in N$ or $bm \in N$. Thus $am' \in f(N)$ or $bm' \in f(N)$. \square

Theorem 2.10. *Let M be an R -module and $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. Suppose that N is a ϕ -classical prime submodule of M . Then*

- (1) *For every $a, b \in R$ and $m \in M$, $(N :_R abm) = (\phi(N) :_R abm) \cup (\phi(N) :_R am) \cup (\phi(N) :_R bm)$;*
- (2) *If R is a u-ring, then for every $a, b \in R$ and $m \in M$, $(N :_R abm) = (\phi(N) :_R abm)$ or $(N :_R abm) = (N :_R am)$ or $(N :_R abm) = (N :_R bm)$*

Proof. (1) Let $a, b \in R$ and $m \in M$. Suppose that $r \in (N :_R abm)$, then $ab(rm) \in N$. If $ab(rm) \in \phi(N)$, then $r \in (\phi(N) :_R abm)$. Therefore we assume that $ab(rm) \notin \phi(N)$. So, either $a(rm) \in N$ or $b(rm) \in N$. Hence, either $r \in (N :_R am)$ or $r \in (N :_R bm)$. Consequently, $(N :_R abm) = (\phi(N) :_R abm) \cup (N :_R am) \cup (N :_R bm)$.

(2) Apply part (1). \square

Let M be an R -module and N a submodule of M . For every $a \in R$, $\{m \in M \mid am \in N\}$ is denoted by $(N :_M a)$. It is easy to see that $(N :_M a)$ is a submodule of M containing N .

In the next theorem we characterize ϕ -classical prime submodules.

Theorem 2.11. *Let M be an R -module, $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ a function and N be a proper submodule of M . The following conditions are equivalent:*

- (1) *N is ϕ -classical prime;*
- (2) *For every $a, b \in R$, $(N :_M ab) = (\phi(N) :_M ab) \cup (N :_M a) \cup (N :_M b)$;*
- (3) *For every $a \in R$ and $m \in M$ with $am \notin N$, $(N :_R am) = (\phi(N) :_R am) \cup (N :_R m)$;*
- (4) *For every $a \in R$ and $m \in M$ with $am \notin N$, $(N :_R am) = (\phi(N) :_R am)$ or $(N :_R am) = (N :_R m)$;*
- (5) *For every $a \in R$ and every ideal I of R and $m \in M$ with $aIm \subseteq N$ and $aIm \not\subseteq \phi(N)$, either $am \in N$ or $Im \subseteq N$;*
- (6) *For every ideal I of R and $m \in M$ with $Im \not\subseteq N$, $(N :_R Im) = (\phi(N) :_R Im)$ or $(N :_R Im) = (N :_R m)$;*
- (7) *For every ideals I, J of R and $m \in M$ with $IJm \subseteq N$ and $IJm \not\subseteq \phi(N)$, either $Im \subseteq N$ or $Jm \subseteq N$.*

Proof. (1) \Rightarrow (2) Suppose that N is a ϕ -classical prime submodule of M . Let $m \in (N :_M ab)$. Then $abm \in N$. If $abm \in \phi(N)$, then $m \in (\phi(N) :_M ab)$. Assume that $abm \notin \phi(N)$. Hence $am \in N$ or $bm \in N$. Therefore $m \in (N :_M a)$ or $m \in (N :_M b)$. Consequently, $(N :_M ab) = (\phi(N) :_M ab) \cup (N :_M a) \cup (N :_M b)$.

(2) \Rightarrow (3) Let $am \notin N$ for some $a \in R$ and $m \in M$. Assume that $x \in (N :_R am)$. Then $axm \in N$, and so $m \in (N :_M ax)$. Since $am \notin N$, then $m \notin (N :_M a)$. Thus by part (2), $m \in (\phi(N) :_M ax)$ or $m \in (N :_M x)$, whence $x \in (\phi(N) :_R am)$ or $x \in (N :_R m)$. Therefore $(N :_R am) = (\phi(N) :_R am) \cup (N :_R m)$.

(3) \Rightarrow (4) By the fact that if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them.

(4) \Rightarrow (5) Let for some $a \in R$, an ideal I of R , $m \in M$, we have that $aIm \subseteq N$ and $aIm \not\subseteq \phi(N)$. Hence $I \subseteq (N :_R am)$ and $I \not\subseteq (\phi(N) :_R am)$. If $am \in N$, then we are done. So, assume that $am \notin N$. Therefore by part (4) we have that $I \subseteq (N :_R m)$, i.e., $Im \subseteq N$.

(5) \Rightarrow (6) \Rightarrow (7) Have proofs similar to that of the previous implications.

(7) \Rightarrow (1) Is trivial. \square

Remark 2.12. Let M be a multiplication R -module and K, L be submodules of M . Then there are ideals I, J of R such that $K = IM$ and $L = JM$. Thus $KL = IJM = IL$. In particular $KM = IM = K$. Also, for any $m \in M$ we define $Km := KRm$. Hence $Km = IRm = Im$.

Theorem 2.13. Let M be a multiplication R -module, N be a proper submodule of M and $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. Then the following conditions are equivalent:

- (1) N is a ϕ -classical prime submodule of M ;
- (2) If $N_1 N_2 m \subseteq N$ for some submodules N_1, N_2 of M and $m \in M$ such that $N_1 N_2 m \not\subseteq \phi(N)$, then either $N_1 m \subseteq N$ or $N_2 m \subseteq N$

Proof. (1) \Rightarrow (2). Let $N_1 N_2 m \subseteq N$ for some submodules N_1, N_2 of M and $m \in M$. Since M is multiplication, there are ideals $I_1, I_2 \in R$ such that $N_1 = I_1 M$ and $N_2 = I_2 M$. Then $N_1 N_2 m = I_1 I_2 m \subseteq N$ and $I_1 I_2 m \not\subseteq \phi(N)$, so by Theorem 2.11, either $I_1 m \subseteq N$ or $I_2 m \subseteq N$. Therefore $N_1 m \subseteq N$ or $N_2 m \subseteq N$.

(2) \Rightarrow (1) Suppose that $I_1 I_2 m \subseteq N$ for some ideals I_1, I_2 of R and some $m \in M$. Then it is sufficient to get $N_1 = I_1 M$ and $N_2 = I_2 M$ in (2). \square

Theorem 2.14. Let R be a um -ring, M be an R -module and N be a proper submodule of M . Suppose that $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. The following conditions are equivalent:

- (1) N is ϕ -classical prime;
- (2) For every $a, b \in R$, $(N :_M ab) = (\phi(N) :_M ab)$ or $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$;
- (3) For every $a, b \in R$ and every submodule L of M , $abL \subseteq N$ and $abL \not\subseteq \phi(N)$ implies that $aL \subseteq N$ or $bL \subseteq N$;
- (4) For every $a \in R$ and every submodule L of M with $aL \not\subseteq N$, $(N :_R aL) = (\phi(N) :_R aL)$ or $(N :_R aL) = (N :_R L)$;
- (5) For every $a \in R$, every ideal I of R and every submodule L of M , $aIL \subseteq N$ and $aIL \not\subseteq \phi(N)$ implies that $aL \subseteq N$ or $IL \subseteq N$;
- (6) For every ideal I of R and every submodule L of M with $IL \not\subseteq N$, $(N :_R IL) = (\phi(N) :_R IL)$ or $(N :_R IL) = (N :_R L)$;
- (7) For every ideals I, J of R and every submodule L of M , $IJL \subseteq N$ and $IJL \not\subseteq \phi(N)$ implies that $IL \subseteq N$ or $JL \subseteq N$.

Proof. (1) \Rightarrow (2) Assume that N is a ϕ -classical prime submodule of M and $a, b \in R$. Then by Theorem 2.11, $(N :_M ab) = (\phi(N) :_M ab) \cup (N :_M a) \cup (N :_M b)$. Since R is a um -ring, then $(N :_M ab) = (\phi(N) :_M ab)$ or $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$.

(2) \Rightarrow (3) Let $abL \subseteq N$ and $abL \not\subseteq \phi(N)$ for some $a, b \in R$ and submodule L of M . Hence $L \subseteq (N :_M ab)$ and $L \not\subseteq (\phi(N) :_M ab)$. Therefore part (2) implies that $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$. So, either $L \subseteq (N :_M a)$ or $L \subseteq (N :_M b)$, i.e., $aL \subseteq N$ or $bL \subseteq N$.

(3) \Rightarrow (4) Let $aL \not\subseteq N$ for some $a \in R$ and submodule L of M . Suppose that $x \in (N :_R aL)$. Then $axL \subseteq N$. If $axL \subseteq \phi(N)$, then $x \in (\phi(N) :_M aL)$. Now, assume that $axL \not\subseteq \phi(N)$. Thus by part (3) we have that $xL \subseteq N$, because $aL \not\subseteq N$. Therefore $x \in (N :_R L)$. Consequently $(N :_R aL) = (\phi(N) :_R aL) \cup (N :_R L)$, and then $(N :_R aL) = (\phi(N) :_R aL)$ or $(N :_R aL) = (N :_R L)$.

(4) \Rightarrow (5) Let for some $a \in R$, an ideal I of R and submodule L of M , we have that

$aIL \subseteq N$ and $aIL \not\subseteq \phi(N)$. Hence $I \subseteq (N :_R aL)$ and $I \not\subseteq (\phi(N) :_R aL)$. If $aL \subseteq N$, then we are done. So, assume that $aL \not\subseteq N$. Therefore by part (4) we have that $I \subseteq (N :_R L)$, i.e., $IL \subseteq N$.

(5) \Rightarrow (6) \Rightarrow (7) Similar to the previous implications.

(7) \Rightarrow (1) Is easy. \square

Theorem 2.15. *Let R be a um-ring, M be an R -module and N be a proper submodule of M . Suppose that $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ and $\psi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be two functions. If N is a ϕ -classical prime submodule of M , then $(N :_R L)$ is a ψ -prime ideal of R for every submodule L of M that is not contained in N with $(\phi(N) :_R L) \subseteq \psi((N :_R L))$.*

Proof. Suppose that N is a ϕ -classical prime submodule of M and L is a submodule of M such that $L \not\subseteq N$, and also $(\phi(N) :_R L) \subseteq \psi((N :_R L))$. Let $ab \in (N :_R L) \setminus \psi((N :_R L))$ for some $a, b \in R$. Then $abL \subseteq N$ and $abL \not\subseteq \phi(N)$. So by Theorem 2.14, $aL \subseteq N$ or $bL \subseteq N$. Hence $a \in (N :_R L)$ or $b \in (N :_R L)$. Consequently $(N :_R L)$ is a ψ -prime ideal of R . \square

Theorem 2.16. *Let R be a um-ring, M be a multiplication R -module and N be a proper submodule of M . Suppose that $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ is a function. Then the following conditions are equivalent:*

- (1) N is a ϕ -classical prime submodule of M ;
- (2) If $N_1N_2N_3 \subseteq N$ for some submodules N_1, N_2, N_3 of M such that $N_1N_2N_3 \not\subseteq \phi(N)$, then either $N_1N_3 \subseteq N$ or $N_2N_3 \subseteq N$;
- (3) If $N_1N_2 \subseteq N$ for some submodules N_1, N_2 of M such that $N_1N_2 \not\subseteq \phi(N)$, then either $N_1 \subseteq N$ or $N_2 \subseteq N$;
- (4) N is a ϕ -prime submodule of M .

Proof. (1) \Rightarrow (2) Let $N_1N_2N_3 \subseteq N$ for some submodules N_1, N_2, N_3 of M such that $N_1N_2N_3 \not\subseteq \phi(N)$. Since M is multiplication, there are ideals I_1, I_2 of R such that $N_1 = I_1M$ and $N_2 = I_2M$. Therefore $N_1N_2N_3 = I_1I_2N_3 \subseteq N$ and $I_1I_2N_3 \not\subseteq \phi(N)$, and so by Theorem 2.14, $I_1N_3 \subseteq N$ or $I_2N_3 \subseteq N$. Hence, $N_1N_3 \subseteq N$ or $N_2N_3 \subseteq N$. (2) \Rightarrow (3) It is obvious.

(3) \Rightarrow (4) Let $r \in R$ and $m \in M$ with $rm \in N \setminus \phi(N)$. Thus $rRm \subseteq N$ but $rRm \not\subseteq \phi(N)$. By Remark 2.12, it follows that $rMRm \subseteq N$. Therefore $rM \subseteq N$ or $Rm \subseteq N$. Hence $r \in (N :_R M)$ or $m \in N$.

(4) \Rightarrow (1) By definition. \square

Corollary 2.17. *Let R be a um-ring, M be a multiplication R -module and N be a proper submodule of M . Suppose that $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ and $\psi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be two functions with $(\phi(N) :_R M) = \psi((N :_R M))$. Then N is a ϕ -prime (ϕ -classical prime) submodule of M if and only if $(N :_R M)$ is a ψ -prime ideal of R .*

Proof. By Theorem 2.15, the “only if” part holds. Suppose that $(N :_R M)$ is a ψ -prime ideal of R . Let $IK \subseteq N$ and $IK \not\subseteq \phi(N)$ for some ideal I of R and some submodule K of M . Since M is multiplication, then there is an ideal J of R such that $K = JM$. Hence $IJ \subseteq (N :_R M)$ and $IJ \not\subseteq \psi((N :_R M))$ which implies that either $I \subseteq (N :_R M)$ or $J \subseteq (N :_R M)$. If $I \subseteq (N :_R M)$, then we are done. So, suppose that $J \subseteq (N :_R M)$. Thus $K = JM \subseteq N$. Consequently N is ϕ -prime. \square

Theorem 2.18. *Let R be a um-ring, M be an R -module and $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function. Suppose that N is a proper submodule of M such that $F \otimes \phi(N) = \phi(F \otimes N)$.*

- (1) *If F is a flat R -module and N is a ϕ -classical prime submodule of M such that $F \otimes N \neq F \otimes M$, then $F \otimes N$ is a ϕ -classical prime submodule of $F \otimes M$.*
- (2) *Suppose that F is a faithfully flat R -module. Then N is a ϕ -classical prime submodule of M if and only if $F \otimes N$ is a ϕ -classical prime submodule of $F \otimes M$.*

Proof. (1) Let $a, b \in R$. Then by Theorem 2.14, either $(N :_M ab) = (\phi(N) :_M ab)$ or $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$. Assume that $(N :_M ab) = (\phi(N) :_M ab)$. Then by [5, Lemma 3.2],

$$\begin{aligned} (F \otimes N :_{F \otimes M} ab) &= F \otimes (N :_M ab) = F \otimes (\phi(N) :_M ab) \\ &= (F \otimes \phi(N) :_{F \otimes M} ab) = (\phi(F \otimes N) :_{F \otimes M} ab). \end{aligned}$$

Now, suppose that $(N :_M ab) = (N :_M a)$. Again by [5, Lemma 3.2],

$$\begin{aligned} (F \otimes N :_{F \otimes M} ab) &= F \otimes (N :_M ab) = F \otimes (N :_M a) \\ &= (F \otimes N :_{F \otimes M} a). \end{aligned}$$

Similarly, we can show that if $(N :_M ab) = (N :_M b)$, then $(F \otimes N :_{F \otimes M} ab) = (F \otimes N :_{F \otimes M} b)$. Consequently by Theorem 2.14 we deduce that $F \otimes N$ is a ϕ -classical prime submodule of $F \otimes M$.

(2) Let N be a ϕ -classical prime submodule of M and assume that $F \otimes N = F \otimes M$. Then $0 \rightarrow F \otimes N \xrightarrow{\subseteq} F \otimes M \rightarrow 0$ is an exact sequence. Since F is a faithfully flat module, $0 \rightarrow N \xrightarrow{\subseteq} M \rightarrow 0$ is an exact sequence. So $N = M$, which is a contradiction. So $F \otimes N \neq F \otimes M$. Then $F \otimes N$ is a ϕ -classical prime submodule by (1). Now for the converse, let $F \otimes N$ be a ϕ -classical prime submodule of $F \otimes M$. We have $F \otimes N \neq F \otimes M$ and so $N \neq M$. Let $a, b \in R$. Then $(F \otimes N :_{F \otimes M} ab) = (\phi(F \otimes N) :_{F \otimes M} ab)$ or $(F \otimes N :_{F \otimes M} ab) = (F \otimes N :_{F \otimes M} a)$ or $(F \otimes N :_{F \otimes M} ab) = (F \otimes N :_{F \otimes M} b)$ by Theorem 2.14. Suppose $(F \otimes N :_{F \otimes M} ab) = (\phi(F \otimes N) :_{F \otimes M} ab)$. Hence

$$\begin{aligned} F \otimes (N :_M ab) &= (F \otimes N :_{F \otimes M} ab) = (\phi(F \otimes N) :_{F \otimes M} ab) \\ &= (F \otimes \phi(N) :_{F \otimes M} ab) = F \otimes (\phi(N) :_M ab). \end{aligned}$$

Thus $0 \rightarrow F \otimes (\phi(N) :_M ab) \xrightarrow{\subseteq} F \otimes (N :_M ab) \rightarrow 0$ is an exact sequence. Since F is a faithfully flat module, $0 \rightarrow (\phi(N) :_M ab) \xrightarrow{\subseteq} (N :_M ab) \rightarrow 0$ is an exact sequence which implies that $(N :_M ab) = (\phi(N) :_M ab)$. With a similar argument we can deduce that if $(F \otimes N :_{F \otimes M} ab) = (F \otimes N :_{F \otimes M} a)$ or $(F \otimes N :_{F \otimes M} ab) = (F \otimes N :_{F \otimes M} b)$, then $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b)$. Consequently N is a ϕ -classical prime submodule of M , by Theorem 2.14. \square

Corollary 2.19. *Let R be a um-ring, M be an R -module and X be an indeterminate. If N is a ϕ -classical prime submodule of M with $R[X] \otimes \phi(N) = \phi(R[X] \otimes N)$, then $N[X]$ is a ϕ -classical prime submodule of $M[X]$.*

Proof. Assume that N is a ϕ -classical prime submodule of M with $R[X] \otimes \phi(N) = \phi(R[X] \otimes N)$. Notice that $R[X]$ is a flat R -module. Then by Theorem 2.18, $R[X] \otimes N \simeq N[X]$ is a ϕ -classical prime submodule of $R[X] \otimes M \simeq M[X]$. \square

Definition 2.20. Let N be a proper submodule of M and $a, b \in R$, $m \in M$. If N is a ϕ -classical prime submodule and $abm \in \phi(N)$, $am \notin N$, $bm \notin N$, then (a, b, m) is called a ϕ -classical triple-zero of N .

Theorem 2.21. Let N be a ϕ -classical prime submodule of an R -module M and suppose that $abK \subseteq N$ for some $a, b \in R$ and some submodule K of M . If (a, b, k) is not a ϕ -classical triple-zero of N for every $k \in K$, then $aK \subseteq N$ or $bK \subseteq N$.

Proof. Suppose that (a, b, k) is not a ϕ -classical triple-zero of N for every $k \in K$. Assume on the contrary that $aK \not\subseteq N$ and $bK \not\subseteq N$. Then there are $k_1, k_2 \in K$ such that $ak_1 \notin N$ and $bk_2 \notin N$. If $abk_1 \notin \phi(N)$, then we have $bk_1 \in N$, because $ak_1 \notin N$ and N is a ϕ -classical prime submodule of M . If $abk_1 \in \phi(N)$, then since $ak_1 \notin N$ and (a, b, k_1) is not a ϕ -classical triple-zero of N , we conclude again that $bk_1 \in N$. By a similar argument, since (a, b, k_2) is not a ϕ -classical triple-zero and $bk_2 \notin N$, then we deduce that $ak_2 \in N$. By our hypothesis, $ab(k_1 + k_2) \in N$ and $(a, b, k_1 + k_2)$ is not a ϕ -classical triple-zero of N . Hence we have either $a(k_1 + k_2) \in N$ or $b(k_1 + k_2) \in N$. If $a(k_1 + k_2) = ak_1 + ak_2 \in N$, then since $ak_2 \in N$, we have $ak_1 \in N$, a contradiction. If $b(k_1 + k_2) = bk_1 + bk_2 \in N$, then since $bk_1 \in N$, we have $bk_2 \in N$, which again is a contradiction. Thus $aK \subseteq N$ or $bK \subseteq N$. \square

Definition 2.22. Let N be a ϕ -classical prime submodule of an R -module M and suppose that $IJK \subseteq N$ for some ideals I, J of R and some submodule K of M . We say that N is a free ϕ -classical triple-zero with respect to IJK if (a, b, k) is not a ϕ -classical triple-zero of N for every $a \in I, b \in J$ and $k \in K$.

Remark 2.23. Let N be a ϕ -classical prime submodule of M and suppose that $IJK \subseteq N$ for some ideals I, J of R and some submodule K of M such that N is a free ϕ -classical triple-zero with respect to IJK . Hence, if $a \in I, b \in J$ and $k \in K$, then $ak \in N$ or $bk \in N$.

Corollary 2.24. Let N be a ϕ -classical prime submodule of an R -module M and suppose that $IJK \subseteq N$ for some ideals I, J of R and some submodule K of M . If N is a free ϕ -classical triple-zero with respect to IJK , then $IK \subseteq N$ or $JK \subseteq N$.

Proof. Suppose that N is a free ϕ -classical triple-zero with respect to IJK . Assume that $IK \not\subseteq N$ and $JK \not\subseteq N$. Then there are $a \in I$ and $b \in J$ with $aK \not\subseteq N$ and $bK \not\subseteq N$. Since $abK \subseteq N$ and N is free ϕ -classical triple-zero with respect to IJK , then Theorem 2.21 implies that $aK \subseteq N$ and $bK \subseteq N$ which is a contradiction. Consequently $IK \subseteq N$ or $JK \subseteq N$. \square

Theorem 2.25. Let M be an R -module and a be an element of R such that $aM \neq M$. Suppose that $(0 :_M a) \subseteq aM$. Then aM is an almost classical prime submodule of M if and only if it is a classical prime submodule of M .

Proof. Assume that aM is an almost classical prime submodule of M . Let $x, y \in R$ and $m \in M$ such that $xym \in aM$. We show that $xm \in aM$ or $ym \in aM$. If $xym \notin (aM :_R M)aM$, then there is nothing to prove, since aM is almost classical prime. So, suppose that $xym \in (aM :_R M)aM$. Note that $(x + a)ym \in aM$. If $(x + a)ym \notin (aM :_R M)aM$, then $(x + a)m \in aM$ or $ym \in aM$. Hence $xm \in aM$ or $ym \in aM$. Therefore assume that $(x + a)ym \in (aM :_R M)aM$. Hence $xym \in (aM :_R M)aM$ gives $aym \in (aM :_R M)aM$. Then, there exists $m' \in (aM :_R M)M$ such that $aym = am'$ and so $ym - m' \in (0 :_M a) \subseteq aM$ which

shows that $ym \in aM$, because $m' \in aM$. Consequently aM is classical prime. The converse is easy to check. \square

Theorem 2.26. *Let M be an R -module and m_0 be an element of M such that $Rm_0 \neq M$. Suppose that $(0 :_R m_0) \subseteq (Rm_0 :_R M)$. If Rm_0 is an almost classical prime submodule of M , then it is a 2-absorbing submodule of M .*

Proof. Assume that Rm_0 is an almost classical prime submodule of M . Let $x, y \in R$ and $m \in M$ such that $xym \in Rm_0$. If $xym \notin (Rm_0 :_R M)m_0$, then there is nothing to prove, since Rm_0 is almost classical prime. So, suppose that $xym \in (Rm_0 :_R M)m_0$. Notice that $xy(m + m_0) \in Rm_0$. If $xy(m + m_0) \notin (Rm_0 :_R M)m_0$, then $x(m + m_0) \in Rm_0$ or $y(m + m_0) \in Rm_0$. Hence $xm \in Rm_0$ or $ym \in Rm_0$. Therefore assume that $xy(m + m_0) \in (Rm_0 :_R M)m_0$. Hence $xym \in (Rm_0 :_R M)m_0$ implies that $xym_0 \in (Rm_0 :_R M)m_0$. Then, there exists $r \in (Rm_0 :_R M)$ such that $xym_0 = rm_0$ and so $xy - r \in (0 :_R m_0) \subseteq (Rm_0 :_R M)$ which shows that $xy \in (Rm_0 :_R M)$. Consequently Rm_0 is a 2-absorbing submodule of M . \square

Proposition 2.27. *Let N be a submodule of M and $\phi(N)$ be a classical prime submodule of M . If N is a ϕ -classical prime submodule of M , then N is a classical prime submodule of M .*

Proof. Let N be a ϕ -classical prime submodule of M . Assume that $abm \in N$ for some elements $a, b \in R$ and $m \in M$. If $abm \in \phi(N)$, then since $\phi(N)$ is classical prime, we conclude that $am \in \phi(N) \subseteq N$ or $bm \in \phi(N) \subseteq N$, and so we are done. When $abm \notin \phi(N)$ clearly the result follows. \square

Let \mathcal{S} be a multiplicatively closed subset of R . It is well-known that each submodule of $\mathcal{S}^{-1}M$ is in the form of $\mathcal{S}^{-1}N$ for some submodule N of M . Let $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function and define $\phi_{\mathcal{S}} : S(\mathcal{S}^{-1}M) \rightarrow S(\mathcal{S}^{-1}M) \cup \{\emptyset\}$ by $\phi_{\mathcal{S}}(\mathcal{S}^{-1}N) = \mathcal{S}^{-1}\phi(N)$ (and $\phi_{\mathcal{S}}(\mathcal{S}^{-1}N) = \emptyset$ when $\phi(N) = \emptyset$) for every submodule N of M .

For an R -module M , the set of zero-divisors of M is denoted by $Z_R(M)$.

Theorem 2.28. *Let M be an R -module, N be a submodule and \mathcal{S} be a multiplicative subset of R .*

- (1) *If N is a ϕ -classical prime submodule of M such that $(N :_R M) \cap \mathcal{S} = \emptyset$, then $\mathcal{S}^{-1}N$ is a $\phi_{\mathcal{S}}$ -classical prime submodule of $\mathcal{S}^{-1}M$.*
- (2) *If $\mathcal{S}^{-1}N$ is a $\phi_{\mathcal{S}}$ -classical prime submodule of $\mathcal{S}^{-1}M$ such that $\mathcal{S} \cap Z_R(N/\phi(N)) = \emptyset$ and $\mathcal{S} \cap Z_R(M/N) = \emptyset$, then N is a ϕ -classical prime submodule of M .*

Proof. (1) Let N be a ϕ -classical prime submodule of M and $(N :_R M) \cap \mathcal{S} = \emptyset$. Suppose that $\frac{a_1}{s_1} \frac{a_2}{s_2} \frac{m}{s_3} \in \mathcal{S}^{-1}N \setminus \phi_{\mathcal{S}}(\mathcal{S}^{-1}N)$ for some $a_1, a_2 \in R$, $s_1, s_2, s_3 \in \mathcal{S}$ and $m \in M$. Then there exists $s \in \mathcal{S}$ such that $sa_1a_2m \in N$. If $sa_1a_2m \in \phi(N)$, then $\frac{a_1}{s_1} \frac{a_2}{s_2} \frac{m}{s_3} = \frac{sa_1a_2m}{ss_1s_2s_3} \in \phi_{\mathcal{S}}(\mathcal{S}^{-1}N) = \mathcal{S}^{-1}\phi(N)$, a contradiction. Since N is a ϕ -classical prime submodule, then we have $a_1(sm) \in N$ or $a_2(sm) \in N$. Thus $\frac{a_1}{s_1} \frac{m}{s_3} = \frac{sa_1m}{ss_1s_3} \in \mathcal{S}^{-1}N$ or $\frac{a_2}{s_2} \frac{m}{s_3} = \frac{sa_2m}{ss_2s_3} \in \mathcal{S}^{-1}N$. Consequently $\mathcal{S}^{-1}N$ is a $\phi_{\mathcal{S}}$ -classical prime submodule of $\mathcal{S}^{-1}M$.

(2) Suppose that $\mathcal{S}^{-1}N$ is a $\phi_{\mathcal{S}}$ -classical prime submodule of $\mathcal{S}^{-1}M$, $\mathcal{S} \cap Z_R(N/\phi(N)) = \emptyset$ and $\mathcal{S} \cap Z_R(M/N) = \emptyset$. Let $a, b \in R$ and $m \in M$ such that $abm \in N \setminus \phi(N)$. Then $\frac{a}{1} \frac{b}{1} \frac{m}{1} \in \mathcal{S}^{-1}N$. If $\frac{a}{1} \frac{b}{1} \frac{m}{1} \in \phi_{\mathcal{S}}(\mathcal{S}^{-1}N) = \mathcal{S}^{-1}\phi(N)$, then there exists $s \in \mathcal{S}$ such that $sabm \in \phi(N)$ which contradicts $\mathcal{S} \cap Z_R(N/\phi(N)) = \emptyset$. Therefore $\frac{a}{1} \frac{b}{1} \frac{m}{1} \in \mathcal{S}^{-1}N \setminus \phi_{\mathcal{S}}(\mathcal{S}^{-1}N)$, and so either $\frac{a}{1} \frac{m}{1} \in \mathcal{S}^{-1}N$ or $\frac{b}{1} \frac{m}{1} \in \mathcal{S}^{-1}N$.

We may assume that $\frac{a}{1} \frac{m}{1} \in \mathcal{S}^{-1}N$. So there exists $u \in \mathcal{S}$ such that $uam \in N$. But $\mathcal{S} \cap Z_R(M/N) = \emptyset$, whence $am \in N$. Consequently N is a ϕ -classical prime submodule of M . \square

Theorem 2.29. *Let N be a ϕ -classical prime submodule of M and suppose that (a, b, m) is a ϕ -classical triple-zero of N for some $a, b \in R, m \in M$. Then*

- (1) $abN \subseteq \phi(N)$.
- (2) $a(N :_R M)m \subseteq \phi(N)$.
- (3) $b(N :_R M)m \subseteq \phi(N)$.
- (4) $(N :_R M)^2m \subseteq \phi(N)$.
- (5) $a(N :_R M)N \subseteq \phi(N)$.
- (6) $b(N :_R M)N \subseteq \phi(N)$.

Proof. (1) Suppose that $abN \not\subseteq \phi(N)$. Then there exists $n \in N$ with $abn \notin \phi(N)$. Hence $ab(m+n) \in N \setminus \phi(N)$, so we conclude that $a(m+n) \in N$ or $b(m+n) \in N$. Thus $am \in N$ or $bm \in N$, which contradicts the assumption that (a, b, m) is ϕ -classical triple-zero. Thus $abN \subseteq \phi(N)$.

(2) Let $axm \notin \phi(N)$ for some $x \in (N :_R M)$. Then $a(b+x)m \notin \phi(N)$, because $abm \in \phi(N)$. Since $xm \in N$, then $a(b+x)m \in N$. Then $am \in N$ or $(b+x)m \in N$. Hence $am \in N$ or $bm \in N$, which contradicts our hypothesis.

(3) The proof is similar to part (2).

(4) Assume that $x_1x_2m \notin \phi(N)$ for some $x_1, x_2 \in (N :_R M)$. Then by parts (2) and (3), $(a+x_1)(b+x_2)m \notin \phi(N)$. Clearly $(a+x_1)(b+x_2)m \in N$. Then $(a+x_1)m \in N$ or $(b+x_2)m \in N$. Therefore $am \in N$ or $bm \in N$ which is a contradiction. Consequently $(N :_R M)^2m \subseteq \phi(N)$.

(5) Let $axn \notin \phi(N)$ for some $x \in (N :_R M)$ and $n \in N$. Therefore by parts (1) and (2) we conclude that $a(b+x)(m+n) \in N \setminus \phi(N)$. So $a(m+n) \in N$ or $(b+x)(m+n) \in N$. Hence $am \in N$ or $bm \in N$. This contradiction shows that $a(N :_R M)N \subseteq \phi(N)$.

(6) Similart to part (5). \square

Theorem 2.30. *If N is a ϕ -classical prime submodule of an R -module M that is not classical prime, then $(N :_R M)^2N \subseteq \phi(N)$.*

Proof. Suppose that N is a ϕ -classical prime submodule of M that is not classical prime. Then there exists a ϕ -classical triple-zero (a, b, m) of N for some $a, b \in R$ and $m \in M$. Assume that $(N :_R M)^2N \not\subseteq \phi(N)$. Hence there are $x_1, x_2 \in (N :_R M)$ and $n \in N$ such that $x_1x_2n \notin \phi(N)$. By Theorem 2.29, $(a+x_1)(b+x_2)(m+n) \in N \setminus \phi(N)$. So $(a+x_1)(m+n) \in N$ or $(b+x_1)(m+n) \in N$. Therefore $am \in N$ or $bm \in N$, a contradiction. \square

Corollary 2.31. *Let M be an R -module and N be a ϕ -classical prime submodule of M such that $\phi(N) \subseteq (N :_R M)^3N$. Then N is ω -classical prime.*

Proof. If N is a classical prime submodule of M , then it is clear. Hence, suppose that N is not a classical prime submodule of M . Therefore by Theorem 2.30 we have $(N :_R M)^2N \subseteq \phi(N) \subseteq (N :_R M)^3N \subseteq (N :_R M)^2N$, that is, $\phi(N) = (N :_R M)^2N = (N :_R M)^3N$. Therefore, $\phi(N) = (N :_R M)^jN$ for all $j \geq 2$ and the result is obtained. \square

As a direct consequence of Theorem 2.30 we have the following result.

Corollary 2.32. *Let M be an R -module and N be a proper submodule of M . If N is an n -almost classical prime submodule ($n \geq 3$) of M that is not classical prime, then $(N :_R M)^2 N = (N :_R M)^{n-1} N$.*

Corollary 2.33. *Let M be a multiplication R -module and N be a proper submodule of M .*

- (1) *If N is a ϕ -classical prime submodule of M that is not classical prime, then $N^3 \subseteq \phi(N)$.*
- (2) *If N is an n -almost classical prime submodule ($n \geq 3$) of M that is not classical prime, then $N^3 = N^n$.*

Proof. (1) Since M is multiplication, then $N = (N :_R M)M$. Therefore by Theorem 2.30 and Remark 2.12, $N^3 = (N :_R M)^2 N \subseteq \phi(N)$.

(2) Notice that $\phi_n(N) = (N :_R M)^{n-1} N = N^n$. Now, use part (1). \square

Theorem 2.34. *Let N be a ϕ -classical prime submodule of M . If N is not classical prime, then*

- (1) $\sqrt{(N :_R M)} = \sqrt{(\phi(N) :_R M)}$.
- (2) *If M is multiplication, then $M\text{-rad}(N) = M\text{-rad}(\phi(N))$.*

Proof. (1) Assume that N is not classical prime. By Theorem 2.30, $(N :_R M)^2 N \subseteq \phi(N)$. Then

$$\begin{aligned} (N :_R M)^3 &= (N :_R M)^2 (N :_R M) \\ &\subseteq ((N :_R M)^2 N :_R M) \\ &\subseteq (\phi(N) :_R M), \end{aligned}$$

and so $(N :_R M) \subseteq \sqrt{(\phi(N) :_R M)}$. Hence, we have $\sqrt{(N :_R M)} = \sqrt{(\phi(N) :_R M)}$.

(2) By part (1), $M\text{-rad}(N) = \sqrt{(N :_R M)M} = \sqrt{(\phi(N) :_R M)M} = M\text{-rad}(\phi(N))$. \square

Theorem 2.35. *Let M be an R -module. Suppose that N_1, N_2 are ϕ -classical prime submodules of M that are not classical prime submodules. Then*

- (1) $\sqrt{(N_1 :_R M) + (N_2 :_R M)} = \sqrt{(\phi(N_1) :_R M) + (\phi(N_2) :_R M)}$.
- (2) *If $N_1 + N_2 \neq M$, $\phi(N_1) \subseteq N_2$ and $\phi(N_2) \subseteq \phi(N_1 + N_2)$, then $N_1 + N_2$ is a ϕ -classical prime submodule.*

Proof. (1) By Theorem 2.34, we have $\sqrt{(N_1 :_R M)} = \sqrt{(\phi(N_1) :_R M)}$ and $\sqrt{(N_2 :_R M)} = \sqrt{(\phi(N_2) :_R M)}$. Now, by [20, 2.25(i)] the result follows.

(2) Suppose that $N_1 + N_2 \neq M$, $\phi(N_1) \subseteq N_2$ and $\phi(N_2) \subseteq \phi(N_1 + N_2)$. Since $(N_1 + N_2)/N_2 \simeq N_1/(N_1 \cap N_2)$ and N_1 is ϕ -classical prime, we get $(N_1 + N_2)/N_2$ is a weakly classical prime submodule of M/N_2 , by Theorem 2.1(3). Now, the assertion follows from Theorem 2.1(4). \square

Theorem 2.36. *Let M_1, M_2 be R -modules and N_1 be a proper submodule of M_1 . Suppose that $\psi_i : S(M_i) \rightarrow S(M_i) \cup \{\emptyset\}$ be functions (for $i = 1, 2$) and let $\phi = \psi_1 \times \psi_2$. Then the following conditions are equivalent:*

- (1) $N = N_1 \times M_2$ is a ϕ -classical prime submodule of $M = M_1 \times M_2$;
- (2) N_1 is a ϕ -classical prime submodule of M_1 and for each $r, s \in R$ and $m_1 \in M_1$ we have

$$rsm_1 \in \psi_1(N_1), \quad rm_1 \notin N_1, \quad sm_1 \notin N_1 \Rightarrow rs \in (\psi_2(M_2) :_R M_2).$$

Proof. (1) \Rightarrow (2) Suppose that $N = N_1 \times M_2$ is a ϕ -classical prime submodule of $M = M_1 \times M_2$. Let $r, s \in R$ and $m_1 \in M_1$ be such that $rs m_1 \in N_1 \setminus \psi_1(N_1)$. Then $rs(m_1, 0) \in N \setminus \phi(N)$. Thus $r(m_1, 0) \in N$ or $s(m_1, 0) \in N$, and so $rm_1 \in N_1$ or $sm_1 \in N_1$. Consequently N_1 is a ϕ -classical prime submodule of M_1 . Now, assume that $rs m_1 \in \psi(N_1)$ for some $r, s \in R$ and $m_1 \in M_1$ such that $rm_1 \notin N_1$ and $sm_1 \notin N_1$. Suppose that $rs \notin (\psi_2(M_2) :_R M_2)$. Therefore there exists $m_2 \in M_2$ such that $rs m_2 \notin \psi_2(M_2)$. Hence $rs(m_1, m_2) \in N \setminus \phi(N)$, and so $r(m_1, m_2) \in N$ and $s(m_1, m_2) \in N$. Thus $rm_1 \in N_1$ or $sm_1 \in N_1$ which is a contradiction. Consequently $rs \in (\psi_2(M_2) :_R M_2)$.

(2) \Rightarrow (1) Let $r, s \in R$ and $(m_1, m_2) \in M = M_1 \times M_2$ be such that $rs(m_1, m_2) \in N \setminus \phi(N)$. First assume that $rs m_1 \notin \psi_1(N_1)$. Then by part (2), $rm_1 \in N_1$ or $sm_1 \in N_1$. So $r(m_1, m_2) \in N$ or $s(m_1, m_2) \in N$, and thus we are done. If $rs m_1 \in \psi_1(N_1)$, then $rs m_2 \notin \psi_2(M_2)$. Therefore $rs \notin (\psi_2(M_2) :_R M_2)$, and so part (2) implies that either $rm_1 \in N_1$ or $sm_1 \in N_1$. Again we have that $r(m_1, m_2) \in N$ or $s(m_1, m_2) \in N$ which shows N is a ϕ -classical prime submodule of M . \square

Proposition 2.37. *Let $R = R_1 \times R_2$ be a decomposable ring, M_1 be an R_1 -module and M_2 be an R_2 -module. Suppose that $\phi : S(M) \rightarrow S(M) \cup \{\emptyset\}$ is a function and $M = M_1 \times M_2$. If N_1 is a weakly classical prime submodule of M_1 satisfying $\{0\} \times M_2 \subseteq \phi(N_1 \times M_2)$, then $N_1 \times M_2$ is a ϕ -classical prime submodule of M .*

Proof. Let $(a, b), (c, d) \in R$ and $(m_1, m_2) \in M$ such that $(a, b)(c, d)(m_1, m_2) \in N_1 \times M_2 \setminus \phi(N_1 \times M_2)$. Then $0 \neq ac m_1 \in N_1$. So $am_1 \in N_1$ or $cm_1 \in N_1$, since N_1 is a weakly classical prime submodule. Hence, $(a, b)(m_1, m_2) \in N_1 \times M_2$ or $(c, d)(m_1, m_2) \in N_1 \times M_2$. \square

Corollary 2.38. *Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times R_2$ be an R -module where M_1 is an R_1 -module. If N_1 is a weakly classical prime submodule of M_1 , then $N = N_1 \times R_2$ is a 3-almost classical prime submodule of M .*

Proof. Suppose that N_1 is a weakly classical prime submodule of M_1 . If N_1 is a classical prime submodule of M_1 , then it is easy to see that N is a classical prime submodule of M and so is ϕ -classical prime submodule of M , for all ϕ . Assume that N_1 is not classical prime. Therefore by 2.30, $(N_1 :_{R_1} M_1)^2 N_1 = \{0\}$ and so $\phi_3(N) = (N :_R M)^2 N = \{0\} \times R_2$. Now, by Proposition 2.37 the result follows. \square

Theorem 2.39. *Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be an R -module where M_1 is an R_1 -module and M_2 is an R_2 -module. Let $\psi_i : S(M_i) \rightarrow S(M_i) \cup \{\emptyset\}$ be functions for $i = 1, 2$ and let $\phi = \psi_1 \times \psi_2$. If $N = N_1 \times M_2$ is a proper submodule of M , then the following conditions are equivalent:*

- (1) N_1 is a classical prime submodule of M_1 ;
- (2) N is a classical prime submodule of M ;
- (3) N is ϕ -classical prime submodule of M where $\psi_2(M_2) \neq M_2$.

Proof. (1) \Rightarrow (2) Let $(a_1, a_2)(b_1, b_2)(m_1, m_2) \in N$ for some $(a_1, a_2), (b_1, b_2) \in R$ and $(m_1, m_2) \in M$. Then $a_1 b_1 m_1 \in N_1$ so either $a_1 m_1 \in N_1$ or $b_1 m_1 \in N_1$ which shows that either $(a_1, a_2)(m_1, m_2) \in N$ or $(b_1, b_2)(m_1, m_2) \in N$. Consequently N is a classical prime submodule of M .

(2) \Rightarrow (3) It is clear that every classical prime submodule is a weakly classical prime submodule.

(3) \Rightarrow (1) Let $abm \in N_1$ for some $a, b \in R_1$ and $m \in M_1$. By assumption, there exists

$m' \in M_2 \setminus \psi_2(M_2)$. Thus $(a, 1)(b, 1)(m, m') \in N \setminus \phi(N)$. So we have $(a, 1)(m, m') \in N$ or $(b, 1)(m, m') \in N$. Thus $am \in N_1$ or $bm \in N_1$. Therefore N_1 is a classical prime submodule of M_1 . \square

Theorem 2.40. *Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be an R -module where M_1 is an R_1 -module and M_2 is an R_2 -module. Let $\psi_i : S(M_i) \rightarrow S(M_i) \cup \{\emptyset\}$ be functions for $i = 1, 2$ where $\psi_2(M_2) = M_2$, and let $\phi = \psi_1 \times \psi_2$. If $N = N_1 \times M_2$ is a proper submodule of M , then N_1 is a ψ_1 -classical prime submodule of M_1 if and only if N is ϕ -classical prime submodule of M .*

Proof. Suppose that N is a ϕ -classical prime submodule of M . First we show that N_1 is a ψ_1 -classical prime submodule of M_1 independently whether $\psi_2(M_2) = M_2$ or $\psi_2(M_2) \neq M_2$. Let $a_1 b_1 m_1 \in N_1 \setminus \psi_1(N_1)$ for some $a_1, b_1 \in R_1$ and $m_1 \in M_1$. Then $(a_1, 1)(b_1, 1)(m_1, m) \in (N_1 \times M_2) \setminus (\psi_1(N_1) \times \psi_2(M_2)) = N \setminus \phi(N)$ for any $m \in M_2$. Since N is a ϕ -classical prime submodule of M , we get either $(a_1, 1)(m_1, m) \in N_1 \times M_2$ or $(b_1, 1)(m_1, m) \in N_1 \times M_2$. So, clearly we conclude that $a_1 m_1 \in N_1$ or $b_1 m_1 \in N_1$. Therefore N_1 is a ψ_1 -classical prime submodule of M_1 . Conversely, suppose that N_1 is ψ_1 -classical prime. Let $(a_1, a_2), (b_1, b_2) \in R$ and $(m_1, m_2) \in M$ such that $(a_1, a_2)(b_1, b_2)(m_1, m_2) \in N \setminus \phi(N)$. Since $\psi_2(M_2) = M_2$, we get $a_1 b_1 m_1 \in N_1 \setminus \psi_1(N_1)$ and this implies that either $a_1 m_1 \in N_1$ or $b_1 m_1 \in N_1$. Thus $(a_1, a_2)(m_1, m_2) \in N$ or $(b_1, b_2)(m_1, m_2) \in N$. \square

Theorem 2.41. *Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be an R -module where M_1 is an R_1 -module and M_2 is an R_2 -module. Suppose that N_1, N_2 are proper submodules of M_1, M_2 , respectively. Let $\psi_i : S(M_i) \rightarrow S(M_i) \cup \{\emptyset\}$ be functions for $i = 1, 2$ and let $\phi = \psi_1 \times \psi_2$. If $N = N_1 \times N_2$ is a ϕ -classical prime submodule of M , then N_1 is a ψ_1 -classical prime submodule of M_1 and N_2 is a ψ_2 -classical prime submodule of M_2 .*

Proof. Suppose that $N = N_1 \times N_2$ is a ϕ -classical prime submodule of M . Let $abm \in N_1 \setminus \psi_1(N_1)$ that $a, b \in R_1$ and $m \in M_1$. Get an element $n \in N_2$. We have $(a, 1)(b, 1)(m, n) \in N \setminus \phi(N)$. Then $(a, 1)(m, n) \in N$ or $(b, 1)(m, n) \in N$. Thus $am \in N_1$ or $bm \in N_1$, and thus N_1 is a ψ_1 -classical prime submodule of M_1 . By a similar argument we can show that N_2 is a ψ_2 -classical prime submodule of M_2 . \square

Theorem 2.42. *Let $R = R_1 \times R_2 \times R_3$ be a decomposable ring and $M = M_1 \times M_2 \times M_3$ be an R -module where M_i is an R_i -module, for $i = 1, 2, 3$. Suppose that $\psi_i : S(M_i) \rightarrow S(M_i) \cup \{\emptyset\}$ be functions such that $\psi(M_i) \neq M_i$ for $i=1,2,3$, and let $\phi = \psi_1 \times \psi_2 \times \psi_3$. If N is a ϕ -classical prime submodule of M , then either $N = \phi(N)$ or N is a classical prime submodule of M .*

Proof. If $N = \phi(N)$, then clearly N is a ϕ -classical prime submodule of M , so we may assume that $N = N_1 \times N_2 \times N_3 \neq \psi_1(N_1) \times \psi_2(N_2) \times \psi_3(N_3)$. Without loss of generality we may assume that $N_1 \neq \psi_1(N_1)$ and so there is $n \in N_1 \setminus \psi_1(N_1)$. We claim that $N_2 = M_2$ or $N_3 = M_3$. Suppose that there are $m_2 \in M_2 \setminus N_2$ and $m_3 \in M_3 \setminus N_3$. Get $r \in (N_2 :_{R_2} M_2)$ and $s \in (N_3 :_{R_3} M_3)$. Since $(1, r, 1)(1, 1, s)(n, m_2, m_3) = (n, rm_2, sm_3) \in N \setminus \phi(N)$, then $(1, r, 1)(n, m_2, m_3) = (n, rm_2, m_3) \in N$ or $(1, 1, s)(n, m_2, m_3) = (n, m_2, sm_3) \in N$. Therefore either $m_3 \in N_3$ or $m_2 \in N_2$, a contradiction. Hence $N = N_1 \times M_2 \times N_3$ or $N = N_1 \times N_2 \times M_3$. Let $N = N_1 \times M_2 \times N_3$. Then $(0, 1, 0) \in (N :_R M)$. Clearly

$(0, 1, 0)^2 N \not\subseteq \psi_1(N_1) \times \psi_2(M_2) \times \psi(N_3)$. So $(N :_R M)^2 N \not\subseteq \phi(N)$ which is a contradiction, by Theorem 2.30. In the case when $N = N_1 \times N_2 \times M_3$ we have that $(0, 0, 1) \in (N :_R M)$ and similar to the previous case we reach a contradiction. \square

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